Mixed stochastic differential equations with long-range dependence: existence, uniqueness and convergence of solutions

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Abstract

For a mixed stochastic differential equation involving standard Brownian motion and an almost surely Hölder continuous process Z with Hölder exponent $\gamma > 1/2$, we establish a new result on its unique solvability. We also establish an estimate for difference of solutions to such equations with different processes Z and deduce a corresponding limit theorem. As a by-product, we obtain a result on existence of moments of a solution to a mixed equation under an assumption that Z has certain exponential moments.

Keywords: Mixed stochastic differential equation, pathwise integral, long-range dependence, fractional Brownian motion, stochastic differential equation with random drift
2010 MSC: 60G22; 60G15; 60H10; 26A33

Introduction

In this paper we study the following mixed stochastic differential equation:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dZ_s,\ t \in [0, T], \quad (1)$$

where W is a standard Wiener process, and Z is an almost surely Hölder continuous process with Hölder exponent $\gamma > 1/2$. The processes W and Z can be dependent.

The motivation to consider such equations comes, in particular, from financial mathematics. When it is necessary to model randomness on a financial market, it is useful to distinguish between two main sources of this randomness. The first source is the stock exchange itself with thousands of agents. The noise

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coming from this source can be assumed white and is best modeled by a Wiener process. The second source has the financial and economical background. The random noise coming from this source usually has a long range dependence property, which can be modeled by a fractional Brownian motion B^H with the Hurst parameter $H > 1/2$ or by a multifractional Brownian motion with the Hurst function uniformly greater than $1/2$. Most of long-range-dependent processes have one thing in common: they are Hölder continuous with exponent greater than $1/2$, and this is the reason to consider a rather general equation (1).

Equation (1) with $Z = B^H$, a fractional Brownian motion, was first considered in [2], where existence and uniqueness of solution was proved for time-independent coefficients and zero drift. For inhomogeneous coefficients, unique solvability was established in [3] for $H \in (3/4, 1)$ and bounded coefficients, in [1] for any $H > 1/2$, but under the assumption that W and B^H are independent, and in [5] for any $H > 1/2$, but bounded coefficient b. In this paper we generalize the last result replacing the boundedness assumption by the linear growth.

There is, however, an obstacle to use equation (1) in applications because it is very hard to analyze with standard tools of stochastic analysis. The main reason for this is that the two stochastic integrals in (1) have very different nature. The integral with respect to the Wiener process is Itô integral, and it is best analyzed in a mean square sense, while the integral with respect to Z is understood in a pathwise sense, and all estimates are pathwise with random constants. So there is a need for good approximations for such equations. One way to approximate is to replace integrals by finite sums, this leads to Euler approximations. For equation (1) such approximations were considered in [4], where a sharp estimate for the rate of convergence was obtained. Another way is to replace process Z by a smooth process $\overline{Z}$, transforming equation (1) into a usual Itô stochastic differential equation with random drift $a(s, x) + c(s, x)Z'_s$. Since there is a well-developed theory for Itô equations, such smooth approximations may prove very useful in applications.

The paper is organized as follows. In Section 1, we give basic facts about integration with respect to fractional Brownian motion and formulate main hypotheses. In Section 2, we establish auxiliary results. As a by-product, we obtain a result on existence of moments of a solution to a mixed equation under an assumption that Z has certain exponential moments, which is satisfied, for example, by a fractional Brownian motion with Hurst parameter $H > 3/4$. Section 3 contains the result about existence and uniqueness of solution to equation (1). In Section 4, we estimate a difference between two solutions of equations (1) with different processes Z and deduce a limit theorem for equation (1) from this estimate.

1. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete probability space equipped with a filtration satisfying standard assumptions, and $\{W_t, t \in [0, T]\}$ be a standard

$\mathcal{F}_t$-Wiener process. Let also $\{Z_t, t \in [0,T]\}$ be an $\mathcal{F}_t$-adapted stochastic process, which is almost surely Hölder continuous with exponent $\gamma > 1/2$. We consider a mixed stochastic differential equation (1). The integral w.r.t. Wiener process W is the standard Itô integral, and the integral w.r.t. Z is pathwise generalized Lebesgue–Stieltjes integral (see [6, 7]), which is defined as follows. Consider two continuous functions f and g, defined on some interval $[a,b] \subset \mathbb{R}$. For $\alpha \in (0,1)$ define fractional derivatives

$$\big(D^{\alpha}_{a+} f\big)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}}du\right)1_{(a,b)}(x),$$

$$\big(D^{1-\alpha}_{b-} g\big)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}}du\right)1_{(a,b)}(x).$$

Assume that $D^{\alpha}_{a+} f \in L_1[a,b]$, $D^{1-\alpha}_{b-} g_{b-} \in L_\infty[a,b]$, where $g_{b-}(x) = g(x) - g(b)$. Under these assumptions, the generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) = e^{i\pi\alpha}\int_a^b \big(D^{\alpha}_{a+} f\big)(x)\big(D^{1-\alpha}_{b-} g_{b-}\big)(x)dx. \tag{2}$$

In view of this, we will consider the following norms for $\alpha \in (1-H, 1/2)$:

$$\|f\|^2_{2,\alpha;t} = \int_0^t \|f\|^2_{\alpha;s}\, g(t,s)ds,$$
$$\|f\|_{\infty,\alpha;t} = \sup_{s\in[0,t]} \|f\|_{\alpha;s}\,,$$

where $g(t,s) = s^{-\alpha} + (t-s)^{-\alpha-1/2}$ and

$$\|f\|_{\alpha;t} = |f(t)| + \int_0^t |f(t)-f(s)|\,(t-s)^{-1-\alpha}ds.$$

Also define a seminorm

$$\|f\|_{0,\alpha;t} = \sup_{0\le u<v<t}\left(\frac{|f(v)-f(u)|}{(v-u)^{1-\alpha}} + \int_u^v \frac{|f(u)-f(z)|}{(z-u)^{2-\alpha}}dz\right).$$

Recall that by our assumption Z is almost surely γ-Hölder continuous with $\gamma > \frac{1}{2}$. Hence it is easy to see that for any $\alpha \in (1-\gamma, 1/2)$

$$\sup_{0\le u<v\le t}\left|\big(D^{1-\alpha}_{v-} Z_{t-}\big)(u)\right| \le \|Z\|_{0,t} < \infty.$$

Thus, we can define the integral with respect to Z by (2), and it admits the following estimate for $0 \le a < b \le t$:

$$\left|\int_a^b f(s)dZ_s\right| \le C_\alpha \|Z\|_{0,t}\int_a^b\left(|f(s)|\,(s-a)^{-\alpha} + \int_a^s |f(s)-f(z)|\,(s-z)^{-\alpha-1}dz\right)ds. \tag{3}$$

for any $\alpha \in (1-\gamma, 1/2)$, $t > 0$, $u \le v \le t$ and any f such that the right-hand side of this inequality is finite.

We will assume that for some $K > 0$ and for any $t, s \in [0,T]$, $x, y \in \mathbb{R}$, $\beta > 1/2$

$$\begin{aligned}
&|a(t,x)| + |b(t,x)| + |c(t,x)| \le K(1+|x|), \quad |\partial_x c(t,x)| \le K,\\
&|a(t,x)-a(t,y)| + |b(t,x)-b(t,y)| + |\partial_x c(t,x) - \partial_x c(t,y)| \le K|x-y|,\\
&|a(s,x)-a(t,x)| + |b(s,x)-b(t,x)| + |c(s,x)-c(t,x)| + |\partial_x c(s,x) - \partial_x c(t,x)| \le K|s-t|^{\beta}.
\end{aligned} \tag{4}$$

It was proved in [5] that equation (1) is uniquely solvable when these assumptions hold and if additionally b is bounded: for some $K_1 > 0$

$$|b(t,x)| \le K_1. \tag{5}$$

The reason for us to formulate this assumption individually is that we are going to drop this assumption.

2. Auxiliary results

Lemma 2.1. *Let $g : [0,T] \to \mathbb{R}$ be a γ-Hölder continuous function. Define for $\varepsilon > 0$ $g^{\varepsilon}(t) = \varepsilon^{-1}\int_{0\vee t-\varepsilon}^{t} g(s)ds$. Then for $\alpha \in (1-\gamma, 1)$*

$$\|g - g^{\varepsilon}\|_{0,\alpha;T} \le CK_{\gamma}(g)\varepsilon^{\gamma+\alpha-1},$$

where $K_{\gamma}(g) = \sup_{0\le s<t\le T} |g(t)-g(s)|/(t-s)^{\gamma}$ is the Hölder constant of g.

Proof. Without loss of generality assume that $g(0) = 0$. To simplify the notation, assume that $g(x) = 0$ for $x < 0$. Take any $t, s \in [0,T]$. For $|t-s| \ge \varepsilon$

$$\begin{aligned}
|g(t) - g^{\varepsilon}(t) - g(s) + g^{\varepsilon}(s)| &= \varepsilon^{-1}\left|\int_{t-\varepsilon}^{t}\big(g(t)-g(u)\big)du - \int_{s-\varepsilon}^{s}\big(g(s)-g(v)\big)dv\right|\\
&\le K_{\gamma}(g)\varepsilon^{-1}\left|\int_{t-\varepsilon}^{t}(t-u)^{\gamma}du\right| + \left|\int_{s-\varepsilon}^{s}(s-v)^{\gamma}dv\right| \le CK_{\gamma}(g)\varepsilon^{\gamma};
\end{aligned}$$

for $|t-s| < \varepsilon$

$$|g(t) - g^{\varepsilon}(t) - g(s) + g^{\varepsilon}(s)| \le |g(t)-g(s)| + \varepsilon^{-1}\left|\int_{-\varepsilon}^{0}\big(g(t+u)-g(s+u)\big)du\right| \le CK_{\gamma}(g)\,|t-s|^{\gamma},$$

consequently

$$|g(t) - g^{\varepsilon}(t) - g(s) + g^{\varepsilon}(s)| \le CK_{\gamma}(g)\big(\varepsilon \wedge |t-s|\,\big)^{\gamma}. \tag{6}$$

Now write

$$\|g - g^{\varepsilon}\|_{0,\alpha;T} \le A^{\varepsilon} + B^{\varepsilon},$$

where

$$
\begin{aligned}
A^\varepsilon &= \sup_{0\le u<v\le T} \frac{|g(u)-g^\varepsilon(u)-g(v)+g^\varepsilon(v)|}{(v-u)^{1-\alpha}} \\
&\le CK_\gamma(g) \sup_{0\le u<v\le T} (v-u)^{\alpha-1}\big(\varepsilon\wedge|v-u|\big)^\gamma \le CK_\gamma(g)\varepsilon^{\gamma+\alpha-1}, \\
B^\varepsilon &= \sup_{0\le u<v\le T} \left|\int_u^v \frac{g(u)-g^\varepsilon(u)-g(x)+g^\varepsilon(x)}{(x-u)^{2-\alpha}}dx\right| \\
&\le CK_\gamma(g) \sup_{0\le u<v\le T} \int_u^v \frac{\big((x-u)\wedge\varepsilon\big)^\gamma}{(x-u)^{2-\alpha}}dx \\
&\le CK_\gamma(g) \sup_{0\le u<v\le T} \big((v-u)\wedge\varepsilon\big)^{\gamma+\alpha-1} \le CK_\gamma(g)\varepsilon^{\gamma+\alpha-1}.
\end{aligned}
$$

□

Corollary 2.1. *Let $g : [0,T] \to \mathbb{R}$ be a γ-Hölder continuous function with $g(0) = 0$. There exists a sequence of continuously differentiable functions $\{g^n, n \ge 1\}$ such that for any $\alpha \in (1-\gamma, 1)$ $\|g-g_n\|_{0,\alpha;T} \to 0$, $n \to \infty$. One possible choice of such sequence is $g_n(t) = a_n^{-1}\int_{0\vee t-a_n}^t g(s)ds$, where $a_n \downarrow 0$, $n \to \infty$.*

Further throughout the paper there will be no ambiguity about α, so for the sake of shortness we will usually abbreviate $\|f\|_t = \|f\|_{\alpha;t}$ and $\|f\|_{x,t} = \|f\|_{x,\alpha;t}$, where $x \in \{0, 2, \infty\}$.

Lemma 2.2. *Under assumptions* (4) *and* (5)

$$
\|X\|_t \le C\|Z\|_{0,t}\left(1+\int_0^t \|X\|_s\left(s^{-\alpha}+(t-s)^{-2\alpha}\right)ds\right)+I_b(t),
$$

where $I_b(t) = \left\|\int_0^\cdot b(s,X_s)dW_s\right\|_t$.

Proof. Write $\|X\|_t \le |X_0| + I_a(t) + I_b(t) + I_c(t)$, where $I_a(t) = \left\|\int_0^\cdot a(s,X_s)ds\right\|_t$, $I_c(t) = \left\|\int_0^\cdot c(s,X_s)dZ_s\right\|_t$. Denote for shortness $\Lambda = \|Z\|_{0,t}$.

Estimate

$$
\begin{aligned}
I_a(t) &\le C\bigg(\int_0^t |a(s,X_s)|\,ds + \int_0^t\int_s^t |a(u,X_u)|\,du(t-s)^{-1-\alpha}ds\bigg) \\
&\le C\bigg(\int_0^t \big(1+|X_s|\big)ds + \int_0^t\int_s^t \big(1+|X_u|\big)du(t-s)^{-1-\alpha}ds\bigg) \\
&\le C\left(1+\int_0^t |X_s|\,ds + \int_0^t |X_u|\,(t-u)^{-\alpha}du\right) \le C\left(1+\int_0^t \|X\|_s\,(t-s)^{-\alpha}ds\right).
\end{aligned}
$$

Further,

$$
I_c(t) \le I_c'(t) + I_c''(t),
$$

where

$$I'_c(t) = \left|\int_0^t c(s, X_s)dZ_s\right| \le C\Lambda \int_0^t \bigg(\big(1 + |X_s|\big)s^{-\alpha} + \int_0^s |X_s - X_u|\,(s-u)^{-1-\alpha}du\bigg)ds$$

$$\le C\Lambda\bigg(1 + \int_0^t \|X\|_s\, s^{-\alpha}ds\bigg),$$

$$I''_c(t) = \int_0^t \left|\int_s^t c(u, X_u)dZ_v\right|(t-s)^{-1-\alpha}ds$$

$$\le C\Lambda \int_0^t \int_s^t \bigg(1 + |X_v|\,(v-s)^{-\alpha} + \int_s^v |X_v - X_z|\,(v-z)^{-1-\alpha}dz\bigg)dv(t-s)^{-1-\alpha}ds$$

$$\le C\Lambda\bigg(1 + \int_0^t \int_s^t \|X\|_v\,(v-s)^{-\alpha}dv(t-s)^{-1-\alpha}ds\bigg)$$

$$= C\Lambda\bigg(1 + \int_0^t \|X\|_v \int_0^v (v-s)^{-\alpha}(t-s)^{-1-\alpha}ds\,dv\bigg) \le C\Lambda\bigg(1 + \int_0^t \|X\|_v\,(t-v)^{-2\alpha}dv\bigg).$$

Combining these estimates, we get

$$\|X\|_t \le C\Lambda\bigg(1 + \int_0^t \|X\|_s\,\big(s^{-\alpha} + (t-s)^{-2\alpha}\big)ds\bigg) + I_b(t).$$

$\square$

Proposition 2.1. *Under assumptions* (4), (5) *and*

$$\mathsf{E}\left[\exp\left\{a\,\|Z\|_{0,T}^{1/(1-2\alpha)}\right\}\right] < \infty, \tag{7}$$

all moments of X are bounded, moreover, $\mathsf{E}\left[\|X\|_{\infty,T}^p\right] < \infty$ *for all* $p > 0$.

Proof. By the generalized Gronwall lemma from [6] it follows from Lemma 2.2 that

$$\|X\|_t \le C\,\|Z\|_{0,t} \sup_{s\in[0,t]} I_b(s)\exp\left\{C\,\|Z\|_{0,t}^{1/(1-2\alpha)}\right\},$$

whence

$$\|X\|_{\infty,T} \le C\,\|Z\|_{0,T} \sup_{s\in[0,T]} I_b(s)\exp\left\{C\,\|Z\|_{0,T}^{1/(1-2\alpha)}\right\},$$

whence the assertion follows, as all moments of $\sup_{s\in[0,T]} I_b(s)$ are bounded due to the Burkholder inequality and the boundedness of b. $\square$

Remark 2.1. The assumption (7) might seem very restrictive. However, it is true if Z is Gaussian and $\alpha < 1/4$ (clearly, such choice of α is possible iff $\gamma > 3/4$). Indeed, it is well known that if supremum of a Gaussian family is finite almost surely, than its square has small exponential moments finite, so we have (7) since $1/(1-2\alpha) < 2$. Examples of such processes are fractional Brownian motion with Hurst parameter $H > 3/4$ and multifractional Brownian motion with Hurst function whose minimal value exceeds $3/4$.

For $N \geq 1$ define a stopping time $\tau_N = \inf\left\{t : \|Z\|_{0,t} \geq N\right\} \wedge T$ and a stopped process $Z_t^N = Z_{t\wedge\tau_N}$, denote by X^N the solution of (1) with Z replaced by Z^N.

Lemma 2.3. *Under assumptions* (4) *and* (5) *it holds*

$$\mathsf{E}\left[\left\|X^N\right\|_{\infty,T}^p\right] < C_{p,N}$$

with the constant $C_{p,N}$ independent of Z and K_1.

Proof. First note that the finiteness of $\mathsf{E}\left[\left\|X^N\right\|_{\infty,T}^p\right]$ can be deduced from Lemma 2.2 exactly the same way as Proposition 2.1.

Second, it follows from Lemma 2.2 and the generalized Gronwall lemma [6] that

$$\left\|X^N\right\|_t \leq CN \sup_{s\in[0,t]} I_b^N(s) \exp\left\{CtN^{1/(1-2\alpha)}\right\} \leq C_N \sup_{s\in[0,t]} I_b^N(s),$$

which implies

$$\left\|X^N\right\|_{\infty,t}^p \leq C_{N,p} \sup_{s\in[0,t]} \left(I_b^N(s)\right)^p.$$

Write

$$\mathsf{E}\left[\sup_{s\in[0,T]} \left(I_b^N(t)\right)^p\right] \leq I_b' + I_b'',$$

where, denoting $b_u^N = b(u, X_u^N)$,

$$I_b' = \mathsf{E}\left[\sup_{s\in[0,t]} \left|\int_0^s b_u^N dW_u\right|^p\right] \leq C_p \mathsf{E}\left[\left(\int_0^t \left|b_s^N\right|^2 ds\right)^{p/2}\right] \leq C_p \mathsf{E}\left[\left(\int_0^t (1+\left\|X^N\right\|_s^2) ds\right)^{p/2}\right]$$

$$\leq C_p\left(1+\mathsf{E}\left[\left(\int_0^t \left\|X^N\right\|_s^2 ds\right)^{p/2}\right]\right) \leq C_p\left(1+\int_0^t \mathsf{E}\left[\left\|X^N\right\|_{\infty,s}^p\right] ds\right),$$

$$I_b'' = \mathsf{E}\left[\sup_{s\in[0,t]} \left(\int_0^s \left|\int_u^s b_z^N dW_z\right| (s-u)^{-1-\alpha} du\right)^p\right].$$

Obviously, we can assume without loss of generality that $p > 4/(1-2\alpha)$. It follows from the Garsia–Rodemich–Rumsey inequality that for arbitrary $\eta \in (0, 1/2-\alpha)$, $u, s \in [0,t]$

$$\left|\int_u^s b_z^N dW_z\right| \leq C\xi_\eta^N(t) \left|s-u\right|^{1/2-\eta}, \quad \xi_\eta^N(t) = \left(\int_0^t \int_0^t \frac{\left|\int_x^y b_v^N dW_v\right|^{2/\eta}}{|x-y|^{1/\eta}} dx\, dy\right)^{\eta/2}.$$

Setting $\eta = 2/p$, we get

$$\xi_\eta^N(t) = \left(\int_0^t \int_0^t \frac{\left|\int_x^y b_v^N dW_v\right|^p}{|x-y|^{p/2}} dx\, dy\right)^{1/p}.$$

Then, similarly to estimate for I_b', we get

$$\begin{aligned}
\mathsf{E}\left[\left(\xi_\eta^N(t)\right)^p\right] &\le C_p \int_0^t\int_0^t \frac{\mathsf{E}\left[\left|\int_x^y b_v^N dW_v\right|^p\right]}{|x-y|^{p/2}} dx\,dy \\
&\le C_p \int_0^t\int_0^y \frac{\mathsf{E}\left[\left(\int_x^y (1+\|X^N\|_{\infty,v}^2) dv\right)^{p/2}\right]}{(y-x)^{p/2}} dx\,dy \\
&\le C_p \int_0^t\int_0^y \frac{(y-x)^{p/2} + \mathsf{E}\left[\left(\int_x^y \|X^N\|_{\infty,v}^2\, dv\right)^{p/2}\right]}{(y-x)^{p/2}} dx\,dy \\
&\le C_p \int_0^t\int_0^y \frac{(y-x)^{p/2} + (y-x)^{p/2-1}\int_x^y \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] dv}{(y-x)^{p/2}} dx\,dy \\
&\le C_p \left(1 + \int_0^t\int_0^y\int_x^y \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] dv (y-x)^{-1} dv\, dx\, dy\right) \\
&= C_p \left(1 + \int_0^t\int_0^y \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] \log\frac{y}{y-v} dv\, dy\right) \\
&\le C_p \left(1 + \int_0^t \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] dv\right).
\end{aligned}$$

whence

$$I_b'' \le C\mathsf{E}\left[\xi_\eta^N(t)^p\right] \sup_{s\in[0,t]} \left(\int_0^t (t-s)^{-1/2-\eta-\alpha} ds\right)^p \le C_p \left(1 + \int_0^t \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] dv\right).$$

Thus, we have the estimate

$$\|X^N\|_{\infty,t}^p \le C_{N,p}\left(1 + \int_0^t \mathsf{E}\left[\|X^N\|_{\infty,v}^p\right] dv\right),$$

from which we derive the desired statement with the help of the Gronwall lemma. □

3. Existence of solution

Now we prove existence and uniqueness of solution to equation (1) without assumption (5). As above, we define a stopped process $Z_t^N = Z_{t\wedge\tau_N}$, where $\tau_N = \inf\left\{t : \|Z\|_{0,t} \ge N\right\} \wedge T$. Denote by X^N the solution of (1) with Z replaced by Z^N.

Theorem 3.1. *If the coefficients of equation* (1) *satisfy conditions* (4)*, then it has a unique solution X such that $\|X\|_{\infty,T} < \infty$ a.s.*

Proof. For integer $N \geq 1$, $R \geq 1$ denote $X^{N,R}$ the solution of equation (1) with process Z replaced by Z^N and coefficient b replaced by $b \wedge (K(R+1)) \vee (-K(R+1))$ (we will call it an (N,R)-equation). Let also $\tau_{N,R} = \inf\left\{t : \left|X_t^{N,R}\right| \geq R\right\} \wedge T$. We argue that $X_t^{N,R'} = X_t^{N,R''}$ a.s. for $t < \tau_{N,R'} \wedge \tau_{N,R''}$.

For brevity define $Y_{t,s} = Y_t - Y_s$ and denote $h(t,s) = (t-s)^{-1-\alpha}$, $1\!\!1_t = 1\!\!1_{t<\tau_{N,R'}\wedge\tau_{N,R''}}$. All the constants in this step may depend on N and R', R''.

Write

$$(X_t^{N,R'} - X_t^{N,R''})1\!\!1_t = \left(\int_0^t a_\Delta(s)ds + \int_0^t b_\Delta(s)dW_s + \int_0^t c_\Delta(s)dZ_s^N\right)1\!\!1_t \tag{8}$$
$$=: (\mathcal{I}_a(t) + \mathcal{I}_b(t) + \mathcal{I}_c(t))\,1\!\!1_t,$$

where $d_\Delta(s) := d(s, X^{N,R'}) - d(s, X^{N,R''}),\ d \in \{a,b,c\}$. Due to our hypotheses, $|d_\Delta(s)| \leq C\left|X_s^{N,R'} - X_s^{N,R''}\right|$.

Define $\Delta_t = \int_0^t \left\|X^{N,R'} - X^{N,R''}\right\|_s^2 1\!\!1_s g(t,s)ds$. Write

$$\Delta_t \leq 6(I'_a + I''_a + I'_b + I''_b + I'_c + I''_c),$$

where $I'_d = \int_0^t \mathcal{I}_d(s)^2 1\!\!1_s g(t,s)ds$, $I''_d = \int_0^t \left(\int_0^s |\mathcal{I}_d(s) - \mathcal{I}_d(u)|h(s,u)du\right)^2 1\!\!1_s g(t,s)ds$, $d \in \{a,b,c\}$.

By the Cauchy–Schwarz inequality, we can write

$$\mathcal{I}_a(s)^2 1\!\!1_s \leq C\int_0^s \left|X_u^{N,R'} - X_u^{N,R''}\right|^2 1\!\!1_u du \leq C\int_0^s \left\|X^{N,R'} - X^{N,R''}\right\|_u^2 1\!\!1_u du, \tag{9}$$

therefore

$$I'_a \leq C\int_0^t \Delta_s\, g(t,s)ds.$$

Similarly,

$$I''_a \leq C\int_0^t \left(\int_0^s \int_u^s \left|X_v^{N,R'} - X_v^{N,R''}\right| dv\, h(s,u)du\right)^2 1\!\!1_s g(t,s)ds$$
$$\leq C\int_0^t \left(\int_0^s \left|X_v^{N,R'} - X_v^{N,R''}\right| 1\!\!1_v (s-v)^{-\alpha}dv\right)^2 g(t,s)ds$$
$$\leq C\int_0^t \int_0^s \left|X_v^{N,R'} - X_v^{N,R''}\right|^2 1\!\!1_v (s-v)^{-\alpha}dv\, g(t,s)ds \leq C\int_0^t \Delta_s\, g(t,s)ds.$$

Further, by (3), for $s \leq t$

$$\mathcal{I}_c(s)^2 1\!\!1_s \leq CN\left[\int_0^s \left(|c_\Delta(u)|u^{-\alpha} + \int_0^u |c_\Delta(u) - c_\Delta(z)|h(u,z)dz\right)du\right]^2 1\!\!1_s$$
$$\leq C\left[\left(\int_0^s |c_\Delta(u)|u^{-\alpha}du\right)^2 + \left(\int_0^s \int_0^u |c_\Delta(u) - c_\Delta(z)|h(u,z)dz\,du\right)^2\right]1\!\!1_s =: C(J'_c + J''_c).$$

Analogously to $\mathcal{I}_a$,

$$J_c' \leq C \int_0^s \left\| X^{N,R'} - X^{N,R''} \right\|_u^2 \mathbb{1}_u u^{-\alpha} du.$$

By Lemma 7.1 of Nualart and Răşcanu (2002), the hypotheses (A)–(D) imply that for any $t_1, t_2, x_1, \dots, x_4$

$$\begin{aligned} &|c(t_1, x_1) - c(t_2, x_2) - c(t_1, x_3) + c(t_2, x_4)| \leq K\,|x_1 - x_2 - x_3 + x_4| \\ &\quad + K\,|x_1 - x_3|\,|t_2 - t_1|^\beta + K\,|x_1 - x_3|\,(|x_1 - x_2| + |x_3 - x_4|). \end{aligned} \tag{10}$$

Therefore,

$$|c_\Delta(u) - c_\Delta(z)| = \left|c(u, X_u^{N,R'}) - c(z, X_z^{N,R'}) - c(u, X_u^{N,R''}) + c(z, X_z^{N,R''})\right|$$

$$\leq C\bigg(\left|X_{u,z}^{N,R'} - X_{u,z}^{N,R''}\right| + \left|X_u^{N,R'} - X_u^{N,R''}\right|(u-z)^\beta + \left|X_u^{N,R'} - X_u^{N,R''}\right|\Big(\left|X_{u,z}^{N,R'}\right| + \left|X_{u,z}^{N,R''}\right|\Big)\bigg).$$

Thus, we have

$$\begin{aligned} J_c'' \leq C\bigg[\int_0^s \int_0^u \Big(&\left|X_{u,z}^{N,R'} - X_{u,z}^{N,R''}\right| + \left|X_u^{N,R'} - X_u^{N,R''}\right|(u-z)^\beta \\ &+ \left|X_u^{N,R'} - X_u^{N,R''}\right|\Big(\left|X_{u,z}^{N,R'}\right| + \left|X_{u,z}^{N,R''}\right|\Big)\Big) h(u,z)\,dz\,\mathbb{1}_u du \bigg]^2 \leq C(H_1 + H_2), \end{aligned}$$

where

$$\begin{aligned} H_1 &= \bigg(\int_0^s \int_0^u \Big(\left|X_{u,z}^{N,R'} - X_{u,z}^{N,R''}\right| + \left|X_u^{N,R'} - X_u^{N,R''}\right|(u-z)^\beta \Big) h(u,z)\,dz\,\mathbb{1}_u du \bigg)^2 \\ &\leq C \int_0^s \Big(\left\|X^{N,R'} - X^{N,R''}\right\|_u \mathbb{1}_u + \left|X_u^{N,R'} - X_u^{N,R''}\right| \mathbb{1}_u u^{\beta-\alpha} \Big)^2 du \leq C \int_0^s \left\|X^{N,R'} - X^{N,R''}\right\|_u^2 \mathbb{1}_u du, \\ H_2 &= \bigg(\int_0^s \left|X_u^{N,R'} - X_u^{N,R''}\right| \int_0^u \Big(\left|X_{u,z}^{N,R'}\right| + \left|X_{u,z}^{N,R''}\right| \Big) h(u,z)\,dz\,\mathbb{1}_u du \bigg)^2 \\ &\leq C\bigg(\int_0^s \left|X_u^{N,R'} - X_u^{N,R''}\right| \left(\left\|X^{N,R'}\right\|_{\infty,u} + \left\|X^{N,R''}\right\|_{\infty,u} \right) \mathbb{1}_u du \bigg)^2 \\ &\leq C(R' + R'')^2 \int_0^s \left|X_u^{N,R'} - X_u^{N,R''}\right|^2 \mathbb{1}_u du \leq C \int_0^s \left\|X^{N,R'} - X^{N,R''}\right\|_u^2 \mathbb{1}_u du. \end{aligned}$$

It follows that

$$\mathcal{I}_c(s)^2 \leq C \int_0^s \left\| X^{N,R'} - X^{N,R''} \right\|_u^2 \mathbb{1}_u u^{-\alpha} du. \tag{11}$$

Consequently,

$$I_c' \leq C \int_0^t \Delta_s\, g(t,s) ds.$$

Now by (3) and (10)

$$I_c'' \le N\int_0^t \left(\int_0^s\int_u^s \left(|c_\Delta(v)|\,(v-u)^{-\alpha} + \int_u^v |c_\Delta(v)-c_\Delta(z)|\,h(v,z)dz\right)dv\, h(s,u)du\right)^2 g(t,s)\mathbb{1}_s ds$$
$$\le C\int_0^t \left(\int_0^s \left(|c_\Delta(v)|\,(s-v)^{-2\alpha} + \int_0^v |c_\Delta(v)-c_\Delta(z)|\,h(v,z)(s-z)^{-\alpha}dz\right)dv\right)^2 g(t,s)\mathbb{1}_s ds$$
$$\le C\int_0^t \left(\int_0^s \left(\big|X_v^{N,R'}-X_v^{N,R''}\big|(s-v)^{-2\alpha} + \int_0^v \left(\big|X_{v,z}^{N,R'}-X_{v,z}^{N,R''}\big| + \big|X_v^{N,R'}-X_v^{N,R''}\big|(v-z)^\beta\right.\right.\right.$$
$$\left.\left.\left. +\big|X_v^{N,R'}-X_v^{N,R''}\big|\big(\big|X_{v,z}^{N,R'}\big| + \big|X_{v,z}^{N,R''}\big|\big)\right)h(v,z)(s-z)^{-\alpha}dz\,\mathbb{1}_v dv\right]^2 g(t,s)ds$$
$$\le C\int_0^t \left[\int_0^s \left(\big|X_v^{N,R'}-X_v^{N,R''}\big|\big((s-v)^{-2\alpha} + (s-v)^{2\beta-2\alpha} + (R'+R'')(s-v)^{-2\alpha}\big)\right.\right.$$
$$\left.\left. + \int_0^v \big|X_{v,z}^{N,R'}-X_{v,z}^{N,R''}\big|h(v,z)dz(s-v)^{-2\alpha}\right)\mathbb{1}_v dv\right]^2 g(t,s)ds \le C\int_0^t \Delta_s\, g(t,s)ds.$$

Summing up and taking expectations, we arrive to

$$\mathsf{E}\left[I_a' + I_a'' + I_c' + I_c''\right] \le C\int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s)ds. \tag{12}$$

Now turn to I_b' and I_b''.

$$\begin{aligned}\mathsf{E}\left[\mathcal{I}_b(s)^2\mathbb{1}_s\right] = \mathsf{E}\left[\left(\int_0^s b_\Delta(u)dW_u\right)^2\mathbb{1}_s\right] &\le \int_0^s \mathsf{E}\left[b_\Delta(u)^2\mathbb{1}_u\right]du\\ &\le C\int_0^s \mathsf{E}\left[(X_u^{N,R'}-X_u^{N,R''})^2\mathbb{1}_u\right]du,\end{aligned} \tag{13}$$

whence

$$\mathsf{E}\left[I_b'\right] \le \int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s)ds.$$

Further,

$$\mathsf{E}\left[I_b''\right] = \int_0^t \mathsf{E}\left[\left(\int_0^s \left|\int_u^s b_\Delta(v)dW_v\right|(s-u)^{-1-\alpha}du\right)^2\mathbb{1}_s\right]g(t,s)ds$$
$$\le \int_0^t\int_0^s \mathsf{E}\left[\left(\int_u^s b_\Delta(v)dW_v\right)^2\mathbb{1}_s\right](s-u)^{-3/2-\alpha}du\int_0^s (s-u)^{-1/2-\alpha}du\, g(t,s)ds$$
$$\le C\int_0^t\int_0^s\int_u^s \mathsf{E}\left[\big|X_v^{N,R'}-X_v^{N,R''}\big|^2\mathbb{1}_v\right]dv(s-u)^{-3/2-\alpha}du\, g(t,s)ds$$
$$\le C\int_0^t\int_0^s \mathsf{E}\left[\big|X_v^{N,R'}-X_v^{N,R''}\big|^2\mathbb{1}_v\right](s-v)^{-1/2-\alpha}dv\, g(t,s)ds \le C\int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s)ds.$$

Combining this with (12), we get

$$\mathsf{E}\left[\Delta_t\right] \le C \int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s)ds,$$

whence we get $\Delta_s = 0$ a.s., which implies $X_t^{N,R'} = X_t^{N,R''}$ for $t < \tau_N \wedge \tau_{N,R'} \wedge \tau_{N,R'}$.

This implies, in particular, that $\tau_{N,R''} \ge \tau_{N,R'}$ a.s. On the other hand, almost surely $\tau_{N,R} = T$ for all R large enough. Indeed, assuming the contrary, for some $t \in [0,T)$ $P(\forall R \ge 1\ \tau_{N,R} < T) = c > 0$ and $\mathsf{E}\left[\left\|X^{R,N}\right\|_\infty\right] \ge cR$, contradicting Lemma 2.3.

It follows that there exists a process $\left\{X_t^N, t\in[0,T]\right\}$ such that for each $R \ge 1$ and $t \le \tau_{N,R}$ $X_t^N = X_t^{N,R}$. Hence, it is evident that X^N solves (1) with Z replaced by Z^N.

Since τ_N increases with N and eventually equals T, we have that there exists a process which solves initial equation (1).

Exactly as above, one can argue that any solution to (1) is a solution to (N,R)-equation for $t < \tau_N \wedge \tau_{N,R}$, which gives uniqueness. □

4. Limit theorem

Let coefficients of (1) satisfy (4), and let X be its unique solution. Let also $\overline{X}$ be the solution to stochastic differential equation

$$\overline{X}_t = X_0 + \int_0^t a(s,\overline{X}_s)ds + \int_0^t b(s,\overline{X}_s)dW_s + \int_0^t c(s,\overline{X}_s)d\overline{Z}_s, \tag{14}$$

where $\overline{Z}$ is a γ-Hölder continuous process.

As above, for $Y \in \left\{Z,\overline{Z}\right\}$ define a stopped process $Y_t^N = Y_{t\wedge\tau_N}$, where $\tau_N = \inf\left\{t : \|Y\|_{0,t} \ge N\right\}\wedge T$, and let X^N and $\overline{X}^N$ be the solutions to corresponding equations. Denote $A_t^{N,R} = \left\{\left\|X^N\right\|_{\infty,t} + \left\|\overline{X}^N\right\|_{\infty,t} \le R\right\}$.

Lemma 4.1. *Under assumptions* (4)*,*

$$\mathsf{E}\left[\left\|X^N - \overline{X}^N\right\|_{2,T}^2 \mathbb{1}_{B_T^{N,R}}\right] \le C_{N,R}\mathsf{E}\left[\left\|Z^N - \overline{Z}^N\right\|_{0,T}\right]$$

with the constant $C_{N,R}$ independent of Z, $\overline{Z}$.

Proof. We will use the same notation as in the proof of Theorem 3.1, except now $\mathbb{1}_t = \mathbb{1}_{A_t^{N,R}}$.

Denote $\Delta_t = \int_0^t \left\|X^N - \overline{X}^N\right\|_s^2 \mathbb{1}_s g(t,s)ds$. Exactly as in the proof of Theorem 3.1 it can be shown that

$$\mathsf{E}\left[\Delta_t\right] \le C\left(C_{N,R}\int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s)ds + \mathsf{E}\left[I'_Z\right] + \mathsf{E}\left[I''_Z\right]\right), \tag{15}$$

where

$$I_Z' = \int_0^t \mathcal{I}_Z(s)^2 g(t,s)\mathbb{1}_s ds,\ I_Z'' = \int_0^t \left(\int_0^s |\mathcal{I}_Z(s) - \mathcal{I}_Z(u)| h(s,u) du\right)^2 g(t,s)\mathbb{1}_s ds,$$

$$\mathcal{I}_Z(t) = \int_0^t c(s, \overline{X}_s) d(Z_s - \overline{Z}_s).$$

By (3), on $A_t^{N,R}$

$$\begin{aligned}
\mathcal{I}_Z(s)^2 &\le C \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 \left(\int_0^s \left(\left| c(u, \overline{X}_u^N) \right| u^{-\alpha} + \int_0^u \left| c(v, \overline{X}_v^N) - c(u, \overline{X}_u^N) \right| h(u,v) dv \right) du \right)^2 \\
&\le C \left\| Z^N - \overline{Z}^N \right\|_{0,T} \left(\int_0^s \left(\left(1 + |\overline{X}_u|\right) u^{-\alpha} + \int_0^u \left((u-v)^\beta + |\overline{X}_u - \overline{X}_v|\right) h(u,v) dv \right) du \right)^2 \\
&\le C \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 \int_0^t \left(1 + \left\|\overline{X}\right\|_{\infty,s}\right)^2 \mathbb{1}_s g(t,s) ds \le C R^2 \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 .
\end{aligned} \tag{16}$$

Hence,

$$I_Z' \le C N^2 \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 .$$

Similarly,

$$\begin{aligned}
I_Z'' \le C \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 \int_0^t \Bigg[\int_0^s \int_u^s \Big(\left| c(v, \overline{X}_v) \right| (v-u)^{-\alpha} \\
+ \int_u^v \left| c(v, \overline{X}_v) - c(z, \overline{X}_z) \right| h(v,z) dz \Big) dv\, h(s,u) du \Bigg]^2 g(t,s) \mathbb{1}_s ds \\
\le C \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 \int_0^t \left[\int_0^s \int_u^s \|X\|_{\infty,v} (v-u)^{-\alpha} dv\, h(s,u) du \right]^2 g(t,s) \mathbb{1}_s ds \le C R^2 \left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 .
\end{aligned}$$

Summing these estimates with (15), we obtain

$$\mathsf{E}\left[\Delta_t\right] \le C_{N,R} \left(\left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 + \int_0^t \mathsf{E}\left[\Delta_s\right] g(t,s) ds \right),$$

whence we get the statement by the generalized Gronwall lemma. □

The proof of the following fact uses the Burkholder inequality and the same ideas as before, so we skip it.

Corollary 4.1. *For $N > 1$ the estimate holds*

$$\mathsf{E}\left[\sup_{t\in[0,T]} \left| X - \overline{X} \right|^2 \mathbb{1}_{A_T^{N,R}} \right] \le C_N \mathsf{E}\left[\left\| Z^N - \overline{Z}^N \right\|_{0,T}^2 \mathbb{1}_{A_T^{N,R}} \right]$$

with the constant C_N independent of Z, $\overline{Z}$.

Finally, we formulate a limit theorem for mixed stochastic differential equation (1).

Let $\{Z^n, n \geq 0\}$ be a sequence of γ-Hölder continuous processes. Consider a sequence of stochastic differential equations

$$X_t^n = X_0 + \int_0^t a(s, X_s^n)ds + \int_0^t b(s, X_s^n)dW_s + \int_0^t c(s, X_s^n)dZ_s^n,\ t \in [0, T]. \quad (17)$$

Theorem 4.1. *Assume that $\|Z - Z^n\|_{0,T} \to 0$ in probability. Then $X_t^n \to X_t$ in probability uniformly in t.*

Proof. Let $B_t^{n,N} = \left\{\|X\|_{\infty,t} + \|X^n\|_{\infty,t} + \|Z\|_{0,t} + \|Z^n\|_{0,t} \leq N\right\}$, $\Delta^n = \sup_{t\in[0,T]} |X_t^n - X_t|$.

For $\varepsilon > 0$ write

$$P(\Delta^n > \varepsilon) \leq P\left(\{\Delta^n > \varepsilon\} \cap B_T^{n,N}\right) + P(\Omega \setminus B_T^{n,N}).$$

From the assumption it is easy to see that $\mathsf{E}\left[\|Z - Z^n\|_{0,T}^2 \mathbb{1}_{B_t^{N,n}}\right] \to 0$, $n \to \infty$. Then by (4.1) we have for any $\varepsilon > 0$

$$P\left(\{\Delta^n > \varepsilon\} \cap B_T^{n,N}\right) \to 0, \quad n \to \infty.$$

So

$$\limsup_{n\to\infty} P(\Delta^n > \varepsilon) \leq \limsup_{n\to\infty} P(\Omega \setminus B_T^{n,N}). \quad (18)$$

We know that $\Lambda_T(Z) < \infty$ a.s., so by assumption, $\|Z^n\|_{0,T}$ are bounded in probability uniformly in n. Therefore by Lemma 2.3, $\|X^n\|_{\infty,T}$ are bounded in probability uniformly in n and $\|X\|_{\infty,T}$ is finite a.s. Consequently, $P(\Omega \setminus B_T^{n,N}) \to 0$, $N \to \infty$ uniformly in n. Thus, we conclude the proof by sending $N \to \infty$ in (18). □

Remark 4.1. Under the assumptions of Theorem 4.1 we have also the convergence in probability $\|X - X^n\|_{2,T} \to 0$, $n \to \infty$.

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